Discrete regular polygons for digital shape rigid motion via polygonization

Phuc Ngo
Yukiko Kenmochi
Nicolas Passat
Isabelle Debled-Rennesson

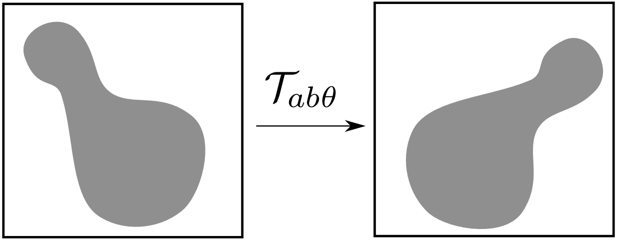
RRPR'18 – 20 August 2018

Rigid motion on $\mathbb{R}^2$

A rigid motion is a bijection defined for $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \mathbb{R}^2$, as

$$\mathcal{T}_{ab\theta}(\mathrm{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

with $a, b \in \mathbb{R}$ and $\theta \in [0, 2\pi[$.

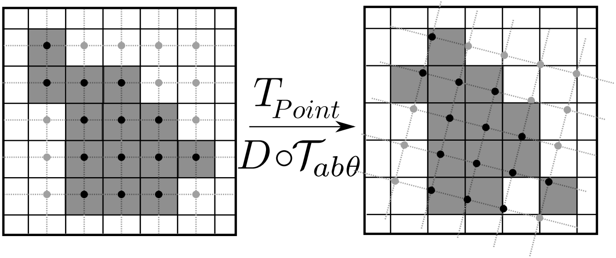


Rigid motion on $\mathbb{Z}^2$

A digital rigid motion on $\mathbb{Z}^2$ is defined for $\mathsf{p} = (\mathsf{p}_1, \mathsf{p}_2) \in \mathbb{Z}^2$ as

$$T_{Point}(\mathsf{p}) = D \circ \mathcal{T}_{ab\theta}(\mathsf{p}) = \begin{pmatrix} [\mathsf{p}_1 \cos\theta - \mathsf{p}_2 \sin\theta + a] \\ [\mathsf{p}_1 \sin\theta + \mathsf{p}_2 \cos\theta + b] \end{pmatrix}$$

where $D : \mathbb{R}^2 \to \mathbb{Z}^2$ is digitization (a rounding function).

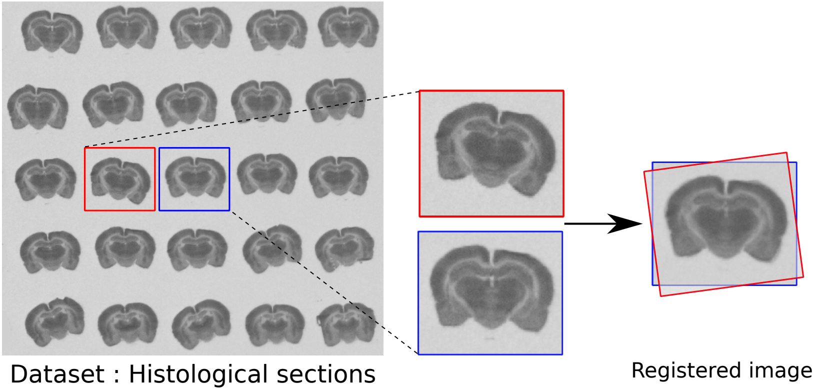


Examples of rigid motions on $\mathbb{Z}^2$: Image registration

Dataset : Histological sections

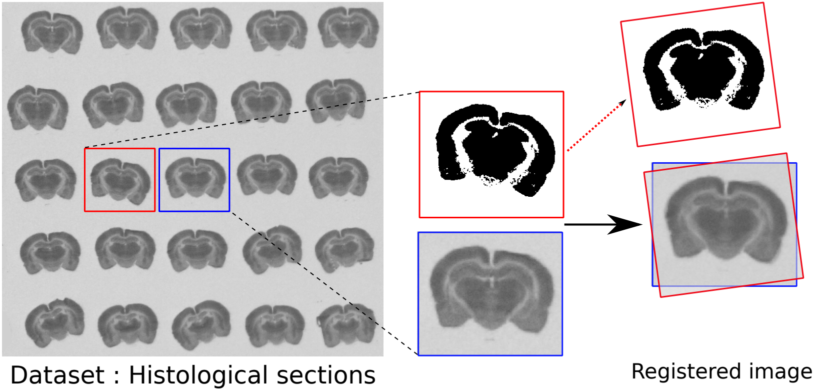
(Laboratory ICube - Strasbourg, France)

Examples of rigid motions on $\mathbb{Z}^2$: Image registration



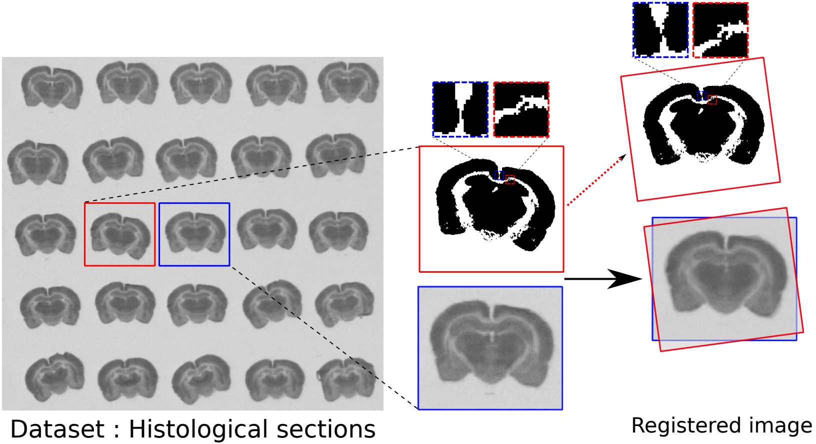
(Laboratory ICube - Strasbourg, France)

Examples of rigid motions on $\mathbb{Z}^2$: Image registration



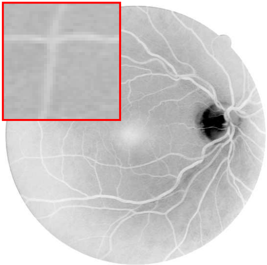
(Laboratory ICube - Strasbourg, France)

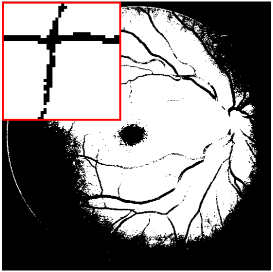
Examples of rigid motions on $\mathbb{Z}^2$: Image registration



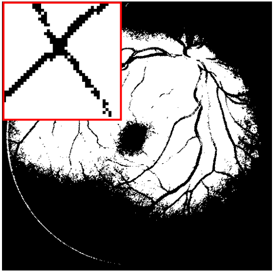
(Laboratory ICube - Strasbourg, France)

Topological and geometrical preservation



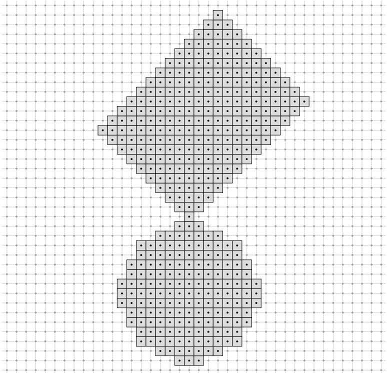


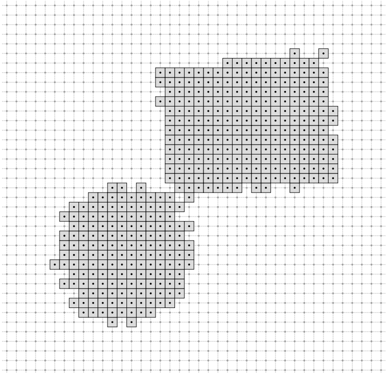
Original image



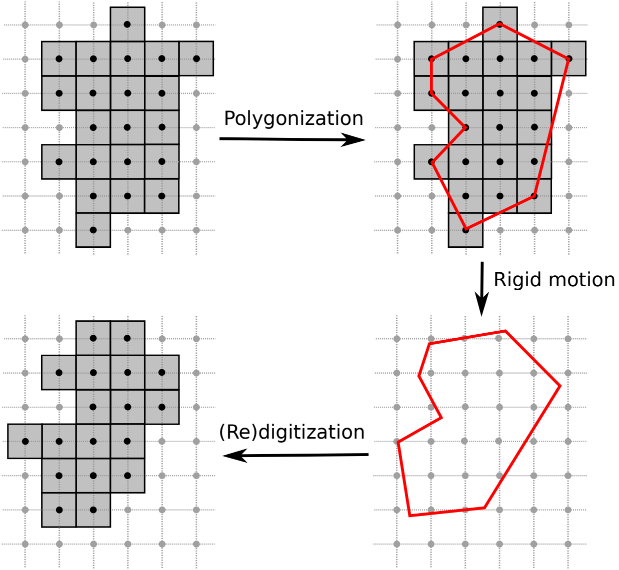
Transformed image

Objective

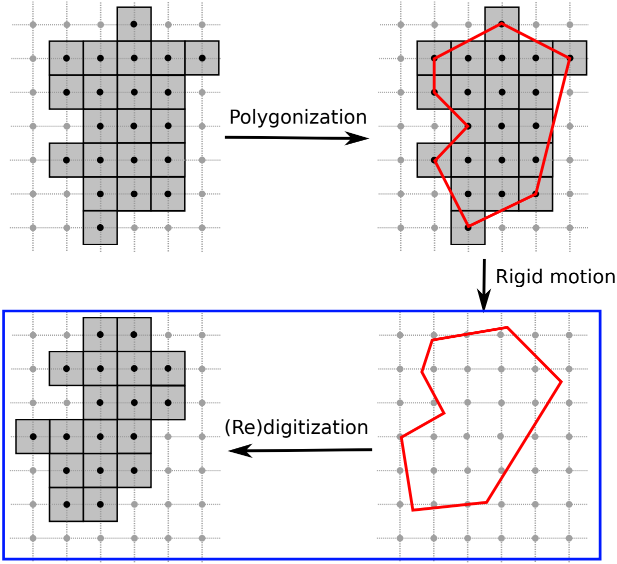




The proposed method



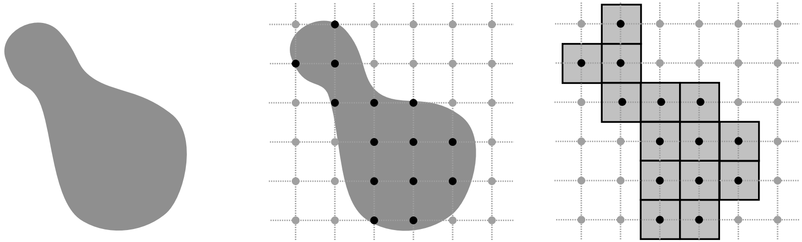
The proposed method



Shape and digitization

Given a finite and connected subset $X \subset \mathbb{R}^2$, its Gauss digitization is defined as :

$$\mathsf{X} = X \cap \mathbb{Z}^2.$$

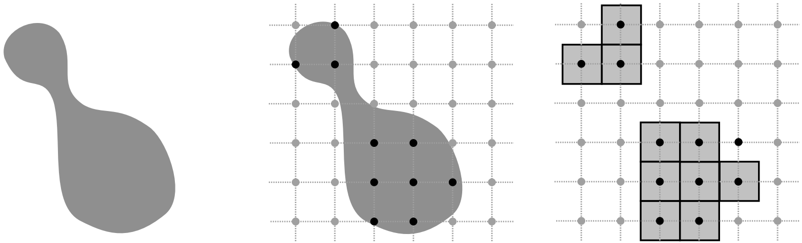


Digitization and topology preservation

Given a finite and connected subset $X \subset \mathbb{R}^2$, its Gauss digitization is defined as:

$$\mathsf{X} = X \cap \mathbb{Z}^2.$$

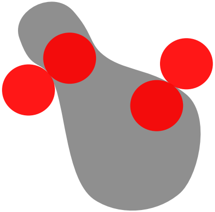
Topology can be altered under the digitization process.

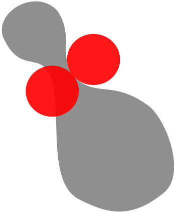


r-regularity

Definition [Pavlidis, 1982]

A finite and connected subset $X \subset \mathbb{R}^2$ is *r-regular* if for each boundary point of X, there exist two tangent open balls of radius r, lying entirely in X and its complement $\overline{X}$, respectively.

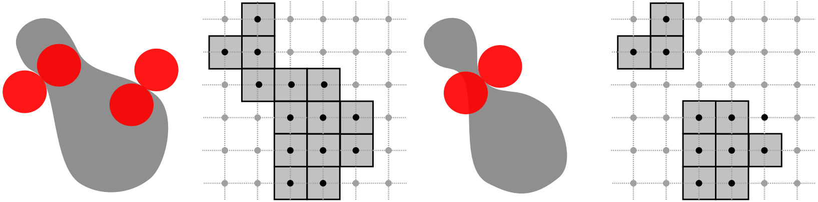




r-regularity for topology preservation

Proposition [Pavlidis, 1982]

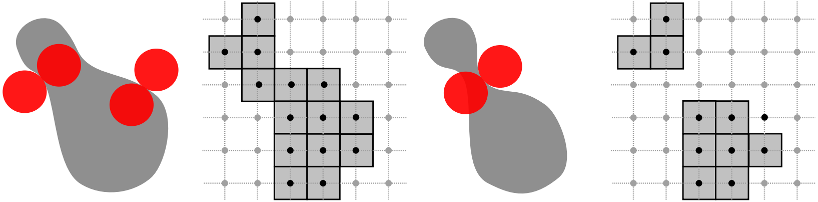
An r-regular set $X \subset \mathbb{R}^2$ has the same topological structure as its digitized version $\mathsf{X} = X \cap \mathbb{Z}^2$ if $r \geq \frac{\sqrt{2}}{2}$.



r-regularity for topology preservation

Proposition [Pavlidis, 1982]

An r-regular set $X \subset \mathbb{R}^2$ has the same topological structure as its digitized version $\mathsf{X} = X \cap \mathbb{Z}^2$ if $r \geq \frac{\sqrt{2}}{2}$.



Objects with non-smooth boundaries (e.g. polygons)?

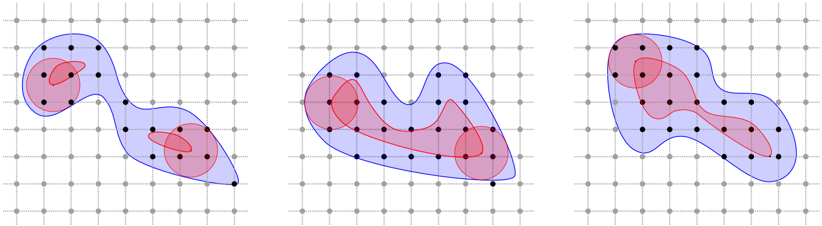
r-regularity

Definition (in Mathematical Morphology)

Let $X \subset \mathbb{R}^2$ be a bounded and simply connected (i.e., connected and wihtout hole) set. If

- $X \ominus B_r$ (rep. $\overline{X} \ominus B_r$) is non-empty and connected, and
- $X = X \ominus B_r \oplus B_r$ (resp. $\overline{X} = \overline{X} \ominus B_r \oplus B_r$)

for a given $r > 0$, we say that X is *r-regular*.



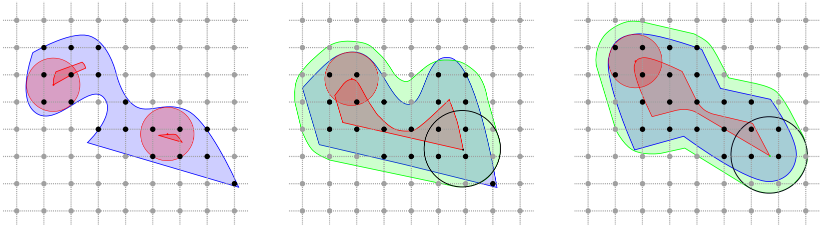
Quasi-r-regularity

Definition [Ngo et al., 2018]

Let $X \subset \mathbb{R}^2$ be a bounded and simply connected set. If

- $X \ominus B_r$ (resp. $\overline{X} \ominus B_r$) is non-empty and connected, and
- $X \subseteq X \ominus B_r \oplus B_{r'}$ (resp. $\overline{X} \subseteq \overline{X} \ominus B_r \oplus B_{r'}$)

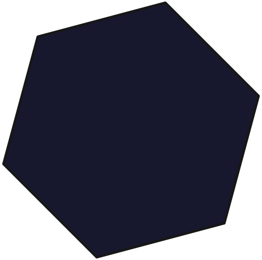
for $r' \geq r > 0$, X is *quasi-r-regular* with "margin" $r' - r$.



Quasi-1-regularity for topology preservation

Proposition [Ngo et al., 2018]

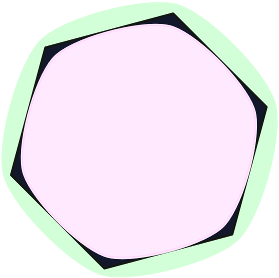
If X is quasi-1-regular with margin $\sqrt{2}-1$, then $\mathsf{X} = X \cap \mathbb{Z}^2$ and $\overline{\mathsf{X}} = \overline{X} \cap \mathbb{Z}^2$ are both 4-connected.



Quasi-1-regularity for topology preservation

Proposition [Ngo et al., 2018]

If X is quasi-1-regular with margin $\sqrt{2}-1$, then $\mathsf{X} = X \cap \mathbb{Z}^2$ and $\overline{\mathsf{X}} = \overline{X} \cap \mathbb{Z}^2$ are both 4-connected.



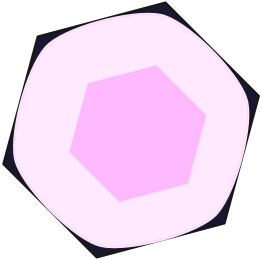
Quasi-1-regularity for topology preservation

Proposition [Ngo et al., 2018]

If X is quasi-1-regular with margin $\sqrt{2}-1$, then $\mathsf{X} = X \cap \mathbb{Z}^2$ and $\overline{\mathsf{X}} = \overline{X} \cap \mathbb{Z}^2$ are both 4-connected.

Idea of proof:

- $X \circ B_1 = X \ominus B_1 \oplus B_1$ is 1-regular, then $(X \circ B_1) \cap \mathbb{Z}^2$ is 4-connected.



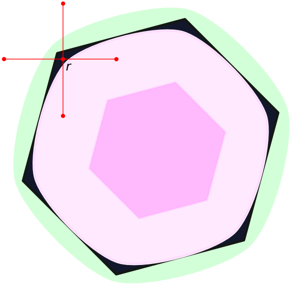
Quasi-1-regularity for topology preservation

Proposition [Ngo et al., 2018]

If X is quasi-1-regular with margin $\sqrt{2}-1$, then $\mathsf{X} = X \cap \mathbb{Z}^2$ and $\overline{\mathsf{X}} = \overline{X} \cap \mathbb{Z}^2$ are both 4-connected.

Idea of proof:

- $X \circ B_1 = X \ominus B_1 \oplus B_1$ is 1-regular, then $(X \circ B_1) \cap \mathbb{Z}^2$ is 4-connected.
- With any position of $\mathbb{Z}^2$, if there exists $\mathsf{r} \in \mathbb{Z}^2$ in $X \setminus (X \circ B_1)$, then r is 4-adjacent to a point of $(X \circ B_1) \cap \mathbb{Z}^2$.



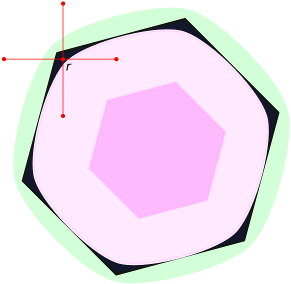
Quasi-1-regularity for topology preservation

Proposition [Ngo et al., 2018]

If X is quasi-1-regular with margin $\sqrt{2}-1$, then $\mathsf{X} = X \cap \mathbb{Z}^2$ and $\overline{\mathsf{X}} = \overline{X} \cap \mathbb{Z}^2$ are both 4-connected.

Idea of proof:

- $X \circ B_1 = X \ominus B_1 \oplus B_1$ is 1-regular, then $(X \circ B_1) \cap \mathbb{Z}^2$ is 4-connected.
- With any position of $\mathbb{Z}^2$, if there exists $\mathsf{r} \in \mathbb{Z}^2$ in $X \setminus (X \circ B_1)$, then r is 4-adjacent to a point of $(X \circ B_1) \cap \mathbb{Z}^2$.



A simple verification of quasi-regularity for polygons is needed.

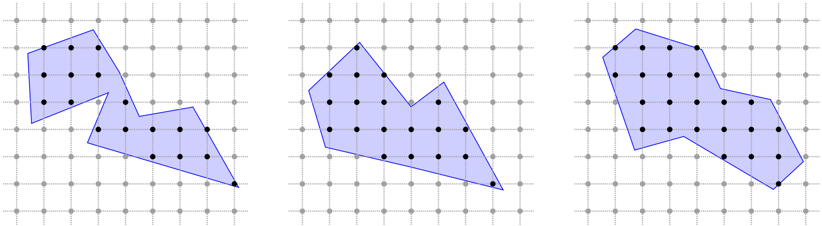
Discrete-1-regularity

Definition [Ngo et al., 2018]

Let P be a simple polygon in $\mathbb{R}^2$, V and E be respectively the set of vertices and edges of P. If P satisfies:

- $\forall v = e_1 \cap e_2 \in V$ with $e_1, e_2 \in E, \forall e \in E \setminus \{e_1, e_2\}, d(v, e) \geq 2$,
- $\forall v = e_1 \cap e_2 \in V$ with $e_1, e_2 \in E, n(e_1).n(e_2) \geq 0$,

then P is *discrete*-1-*regular*.



Discrete-1-regularity and quasi-1-regularity

Proposition [Ngo et al., 2018]

Let $P \subset \mathbb{R}^2$ be a simple polygon. If P is discrete-1-regular, then P is quasi-1-regular.

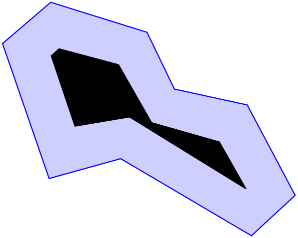
Discrete-1-regularity and quasi-1-regularity

Proposition [Ngo et al., 2018]

Let $P \subset \mathbb{R}^2$ be a simple polygon. If P is discrete-1-regular, then P is quasi-1-regular.

Idea of proof:

- $d(v, e) \geq 2$, thus $P \ominus B_1$ is non-empty and connected.



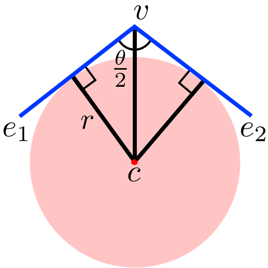
Discrete-1-regularity and quasi-1-regularity

Proposition [Ngo et al., 2018]

Let $P \subset \mathbb{R}^2$ be a simple polygon. If P is discrete-1-regular, then P is quasi-1-regular.

Idea of proof:

- $d(v, e) \geq 2$, thus $P \ominus B_1$ is non-empty and connected.
- $n(e_1).n(e_2) \geq 0 \implies \frac{\sqrt{2}}{2} \leq \sin\frac{\theta}{2} \leq 1$. Since $\sin\frac{\theta}{2} = \frac{1}{d(c,v)}$, $r \leq d(c, v) \leq \sqrt{2}$. Thus $X \subseteq X \ominus B_1 \oplus B_{\sqrt{2}}$.



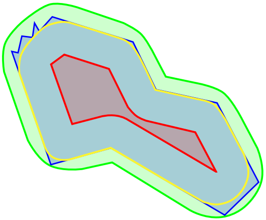
Discrete-1-regularity and quasi-1-regularity

Proposition [Ngo et al., 2018]

Let $P \subset \mathbb{R}^2$ be a simple polygon. If P is discrete-1-regular, then P is quasi-1-regular.

Discrete-1-regular objects is a subset of quasi-1-regular objects for polygons

- Smooth boundary
- Noisy boundary



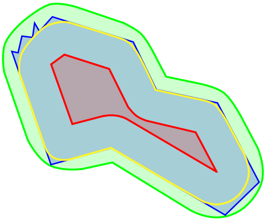
Discrete-1-regularity and quasi-1-regularity

Proposition [Ngo et al., 2018]

Let $P \subset \mathbb{R}^2$ be a simple polygon. If P is discrete-1-regular, then P is quasi-1-regular.

Discrete-1-regular objects is a subset of quasi-1-regular objects for polygons

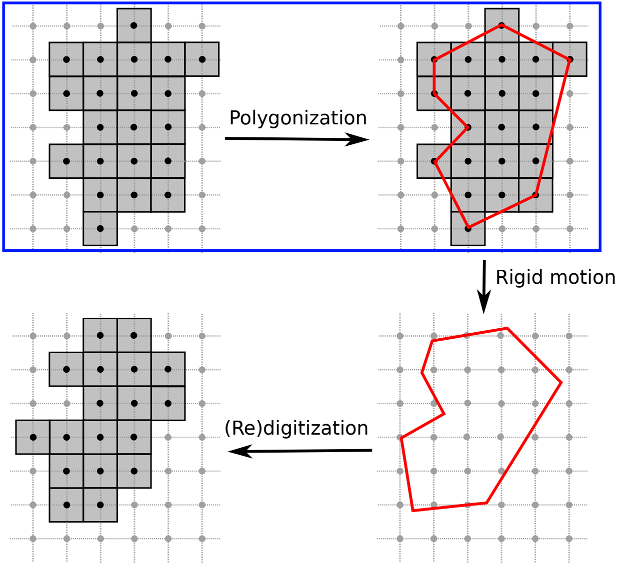
- Smooth boundary
- Noisy boundary



Lemma [Ngo et al., 2018]

If P is discrete-1-regular, then $P \cap \mathbb{Z}^2$ is 4-connected.

The proposed method

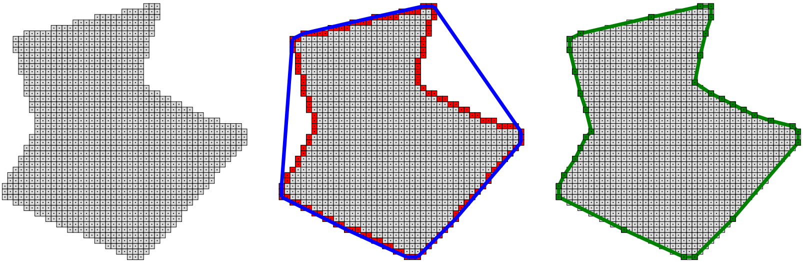


Polygonization of digital objects

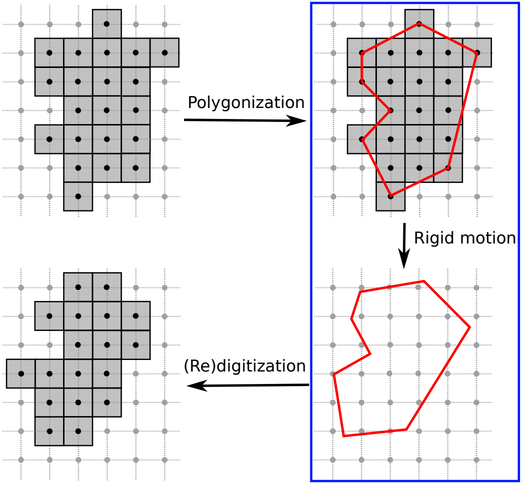
The method is based on contour points and the convex hull

1. Extract 8-connected contour points of X
2. Compute convex hull of X
3. Find the segments that fit concave parts of X with reversibility:

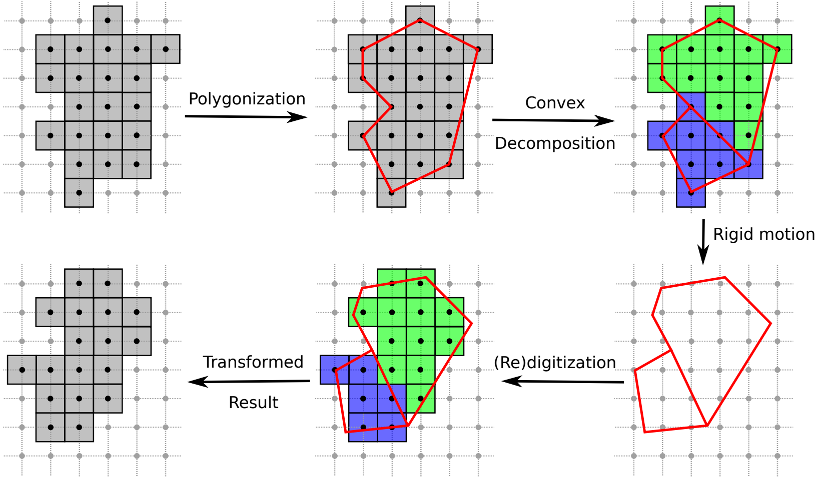
$$\mathsf{X} = P(\mathsf{X}) \cap \mathbb{Z}^2$$



Rigid motion with convex decomposition



Rigid motion with convex decomposition

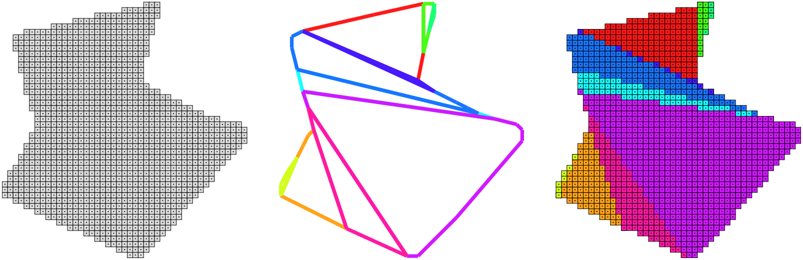


Convex decomposition of polygons

The method [Lien and Amato, 2006] decomposes a simple polygon into convex pieces by iteratively removing the most significant non-convex features.

$$P = \bigcup P_i$$

$$\mathsf{X} = P(\mathsf{X}) \cap \mathbb{Z}^2 = \bigcup \left(P_i \cap \mathbb{Z}^2\right).$$

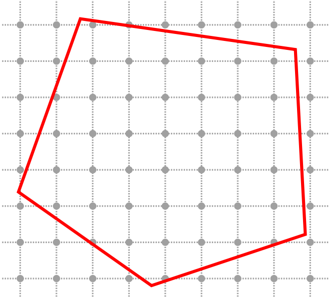


Digitization of convex object using H-representation

Each decomposed convex polygon P_i is represented by a set of half-planes $\mathcal{R}(P_i)$ such that

$$P_i \cap \mathbb{Z}^2 = \Big(\bigcap_{\mathrm{H} \in \mathcal{R}(P_i)} \mathrm{H} \Big) \cap \mathbb{Z}^2$$

where $\mathcal{R}(P_i)$ is the minimal set of half-planes that include P_i.

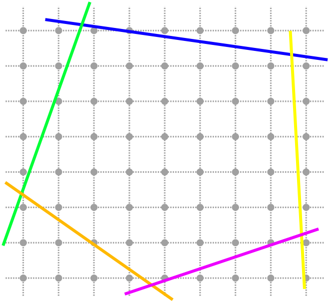


Digitization of convex object using H-representation

Each decomposed convex polygon P_i is represented by a set of half-planes $\mathcal{R}(P_i)$ such that

$$P_i \cap \mathbb{Z}^2 = \Big(\bigcap_{\mathrm{H} \in \mathcal{R}(P_i)} \mathrm{H} \Big) \cap \mathbb{Z}^2$$

where $\mathcal{R}(P_i)$ is the minimal set of half-planes that include P_i.

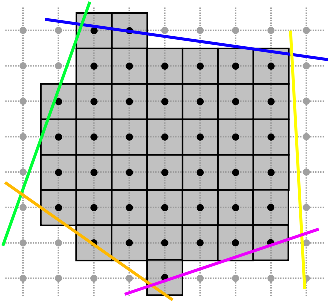


Digitization of convex object using H-representation

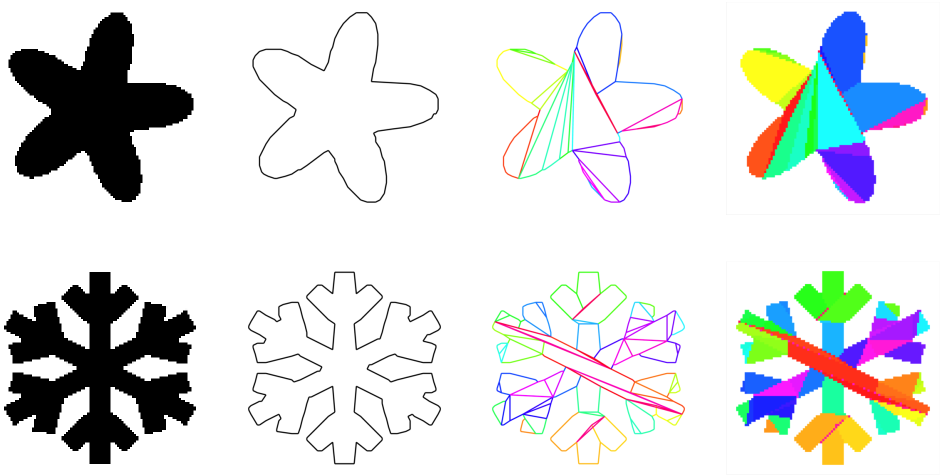
Each decomposed convex polygon P_i is represented by a set of half-planes $\mathcal{R}(P_i)$ such that

$$P_i \cap \mathbb{Z}^2 = \Big(\bigcap_{\mathrm{H} \in \mathcal{R}(P_i)} \mathrm{H} \Big) \cap \mathbb{Z}^2$$

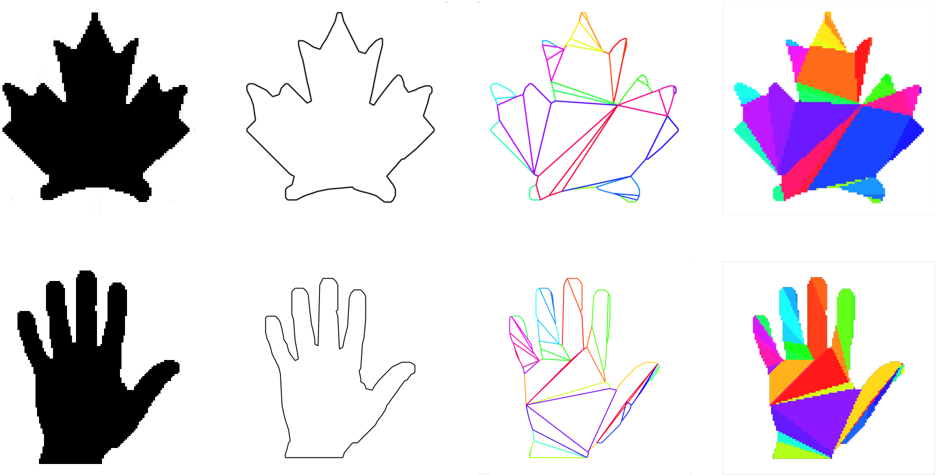
where $\mathcal{R}(P_i)$ is the minimal set of half-planes that include P_i.



Convex decomposition of polygons



Convex decomposition of polygons



Topological preserving rigid motions

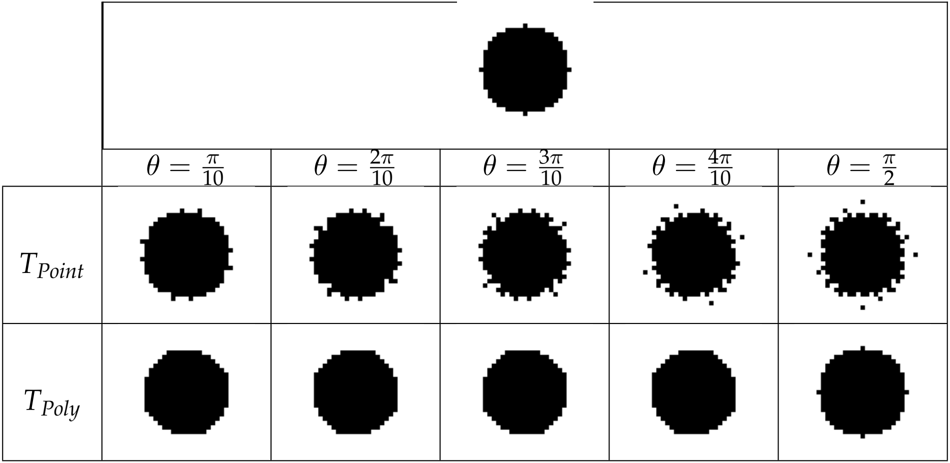
Rigid motion with convex decomposition

$$T_{\mathcal{P}oly}(\mathsf{X}) = \mathcal{T}(\mathcal{P}oly(\mathsf{X})) \cap \mathbb{Z}^2$$

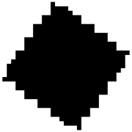
Proposition [Ngo et al., 2018]

Let $\mathsf{X} \subset \mathbb{Z}^2$ be a digital object. Let $P(\mathsf{X}) \subset \mathbb{R}^2$ be a polygon such that $P(\mathsf{X}) \cap \mathbb{Z}^2 = \mathsf{X}$. If $P(\mathsf{X})$ is discrete-1-regular, then $T_{\mathcal{P}oly}(\mathsf{X})$ is 4-connected.

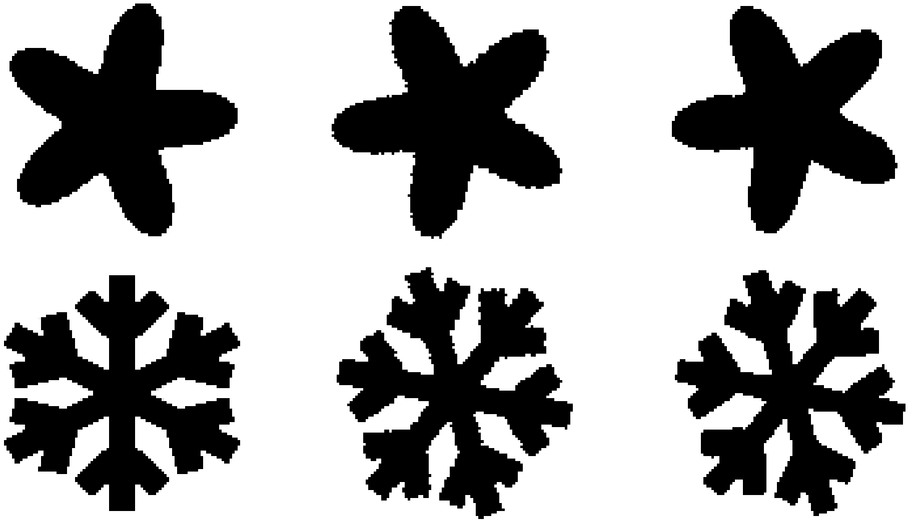
Experimental results



Experimental results

	$\theta = \frac{\pi}{10}$	$\theta = \frac{2\pi}{10}$	$\theta = \frac{3\pi}{10}$	$\theta = \frac{4\pi}{10}$	$\theta = \frac{\pi}{2}$
T_{Point}					
T_{Poly}					

Experimental results



X $\qquad T_{Point}(\mathsf{X})$ $\qquad T_{\mathcal{P}oly}(\mathsf{X})$

Experimental results

X $T_{Point}(X)$ $T_{\mathcal{P}oly}(X)$

Code sources

Code sources are available at the github repository:
`https://github.com/ngophuc/RigidTransformAcd2D`

- Compilation avec Cmake[1]
- Dependence : DGtal library[2]
- Code packages:
  - **polygonization** computes the polygon from a digital image
  - **decomposeShapeAcd2d** decomposes a polygon into the convex parts using the ACD method[3]
  - **transformAConvexShape** implements the proposed rigid motion method.
- Examples on github repository webpage

[1] `https://cmake.org/`
[2] `https://dgtal.org`
[3] `https://github.com/jmlien/acd2d`

Online demonstration

An online demonstration based on the DGtal library, is available at the following website:
`http://ipol-geometry.loria.fr/~phuc/ipol_demo/RigidMotion2D`

Rigid Motion of Quasi Regular Object: Online Demonstration

article | demo | archive

Please cite the reference article if you publish results obtained with this online demo.

This demonstration applies the Rigid Motion on Quasi Regular Objects.

Select Data

Click on an image to use it as the algorithm input.

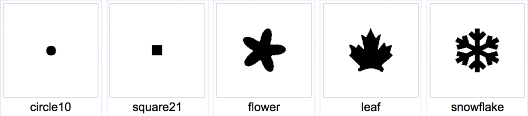


image credits

Upload 2D Images

Upload your **2D binary image** to use as the algorithm input. Note that the algorithm handles only **a well-composed object** in the image.

input image | Choose file | No file chosen | upload

Images larger than 16777216 pixels will be resized. Upload size is limited to 16MB per image file and 10MB for the whole upload set .
PNG format is supported. The uploaded will be publicly archived unless you switch to private mode on the result page.
Only upload suitable images. See the copyright and legal conditions for details.

Conclusion

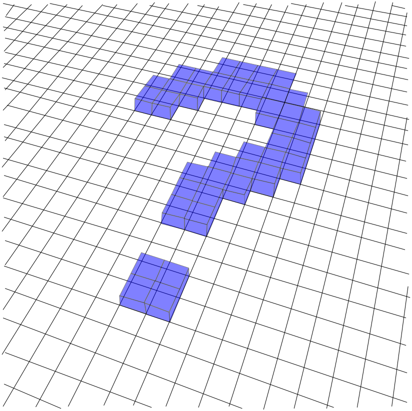
Contributions:

- A notion of *discrete*-1-*regularity* for polygonal objects, as a subset of *quasi*-1-*regular* objects, that provides sufficient conditions for topology preservation by Gaussian digitization.
- A rigid motion scheme based on polygonal representation that preserves geometry and topology properties of the transformed digital object.

Perspectives:

- Necessary conditions for topology and geometry preservation.
- A polygonization method providing discrete-regular polygons of digital objects.
- Regularization method for non discrete-regular polygons.

Thank you for your attention!



References I

Kim, C. E. (1981).
On the cellular convexity of complexes.
IEEE Transactions on Pattern Analysis and Machine Intelligence, PAMI-3(6):617–625.

Latecki, L. J., Conrad, C., and Gross, A. (1998).
Preserving topology by a digitization process.
Journal of Mathematical Imaging and Vision, 8(2):131–159.

Latecki, L. J., Eckhardt, U., and Rosenfeld, A. (1995).
Well-composed sets.
Computer Vision and Image Understanding, 61(1):70–83.

References II

Lien, J.-M. and Amato, N. M. (2006).
Approximate convex decomposition of polygons.
Comput. Geom. Theory Appl., 35(1-2):100–123.

Ngo, P., Passat, N., Kenmochi, Y., and Talbot, H. (2014).
Topology-preserving rigid transformation of 2D digital images.
IEEE Transactions on Image Processing, 23(2):885–897.

Pavlidis, T. (1982).
Algorithms for graphics and image processing.
Berlin: Springer, and Rockville: Computer Science Press.

Extension to 3D

Definition

Let $X \subset \mathbb{R}^3$ be a bounded, simply connected set. If

- $X \ominus B_r$ (resp. $\overline{X} \ominus B_r$) is non-empty and connected, and
- $X \subseteq X \ominus B_r \oplus B_{r'}$ (resp. $\overline{X} \subseteq \overline{X} \ominus B_r \oplus B_{r'}$)

for $r' \geq r > 0$, X is *quasi-r-regular* with "margin" $r' - r$.

Proposition

Let $\mathsf{X} \subset \mathbb{Z}^3$ be a digital object. If X is quasi-1-regular with margin $\frac{2}{\sqrt{3}} - 1$, then $\mathsf{X} = X \cap \mathbb{Z}^3$ and $\overline{\mathsf{X}} = \overline{X} \cap \mathbb{Z}^3$ are both 6-connected.

Experimental results